Mark Scheme Midterm Exam Calculus 1

23 september 2013, 9.00-11.00.

Write on each page your name and student number, and on the first page your seminar group. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by arguments/work. Success.

miscalculation = -0.1

1. (a) Formulate the principle of mathematical induction.

   0.2: (method of) proof;
   0.2: statement for all positive integers;
   0.2: base case (n=1);
   0.4: inductive step (0.2 for the induction hypothesis).

   (b) Prove that if $n \geq 1$ is a positive integer, then

   $$2^1 + 2^2 + \cdots + 2^n \;=\; 2^{n+1} - 2$$

   0.2: principle of mathematical induction;
   0.4: verification statement for n=1: $2 = 4 - 2$*;*
   0.3: induction hypothesis: suppose

   $$2^1 + 2^2 + \cdots + 2^k \;=\; 2^{k+1} - 2$$

   0.1: where k is an arbitrary positive integer
   0.2: it is to be shown that

   $$2^1 + 2^2 + \cdots + 2^{k+1} \;=\; 2^{k+2} - 2$$

   0.2: start proof of induction step

   $$(2^1 + 2^2 + \cdots 2^k) + 2^{k+1} \;=$$

   0.4: use inductive hypothesis

   $$2^{k+1} - 2 + 2^{k+1} \;=$$

   0.2: end of proof

   $$2\,2^{k+1} - 2 \;=\; 2^{k+2} - 2$$

2. Prove that for all integers $n \geq 1$,
$$8^n - 3^n$$
is divisible by 5.

0.2: principle of mathematical induction;
0.2: verification statement for n=1: $(8-3)/5 = 1$;
0.2: induction hypothesis: suppose that
$$8^k - 3^k$$
0.1: is divisible by 5, where k *is an arbitrary positive integer*
0.2: it is to be shown that
$$8^{k+1} - 3^{k+1}$$
is divisible by 5
0.1: start proof of induction step
$$8^{k+1} - 3^{k+1} = 8\,8^k - 3^{k+1}$$
0.3:
$$8^{k+1} - 3^{k+1} = 8\,(8^k - 3^k) + 8\,3^k - 3^{k+1}$$
0.1
$$8^{k+1} - 3^{k+1} = 8\,(8^k - 3^k) + (8-3)\,3^k$$
0.3: use inductive hypothesis
$$(8^k - 3^k)$$
is divisible by 5
0.3:
$$(8-3)\,3^k = 5\,3^k$$
is also divisible by 5; hence $8^{k+1} - 3^{k+1}$ *is divisible by 5.*

3. Solve
$$z^2 = \frac{1}{1-i}$$
and sketch the solutions in the complex plane.

0.2: idea polar form $z = re^{i\theta}$
0.2: $z^2 = r^2 e^{i2\theta}$
0.2:
$$|\frac{1}{1-i}| = \frac{|1|}{|1-i|} = \frac{1}{\sqrt{2}}$$
0.2:
$$arg(\frac{1}{1-i}) = arg(1) - arg(1-i) = 0 - -\pi/4 + 2k\pi$$
0.1: with k *any integer*
0.2: hence 1: $r^2 = 1/\sqrt{2}$
0.2: hence 2: $2\theta = \pi/4 + 2k\pi$

0.2: solutions: $r = \sqrt{1/\sqrt{2}}$ *and* $\theta = \pi/8 + k\pi$
0.1: two solutions corresponding to $k = 0$ *and* $k = 1$*, e.g.*
0.4: sketch (-0.2 if radius is unclear; -0.2 if angles are unclear)

4. Determine all complex numbers z satisfying

$$\mathrm{e}^{iz} = -1$$

0.2: idea $z = x + iy$
0.2: $\mathrm{e}^{iz} = \mathrm{e}^{-y}\,\mathrm{e}^{ix}$
0.2: idea polar form
0.2: $|\mathrm{e}^{iz}| = \mathrm{e}^{-y}$
0.2: $arg(\mathrm{e}^{iz}) = x$
0.2: $-1 = 1\mathrm{e}^{i(\pi+2k\pi)}$
0.2: with k *any integer*
0.2: hence 1: $\mathrm{e}^{-y} = 1$
0.2: hence 2: $x = \pi + 2k\pi$
0.2: solutions: $z = \pi + 2k\pi$

Maximum score:

1a	1.0	2	2.0	3	2.0	4	2.0
b	2.0						

Total: 9 +1 (free) = 10.